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Consolidity analysis for fully fuzzy functions, matrices, probability and statistics

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Abstract The paper presents a *comprehensive review* of the know-how for developing the systems ***consolidity theory*** for modeling, analysis, optimization and design in ***fully fuzzy environment***. The solving of systems consolidity theory included its development for handling new functions of different dimensionalities, fuzzy analytic geometry, fuzzy vector analysis, functions of fuzzy complex variables, ordinary differentiation of fuzzy functions and partial fraction of fuzzy polynomials. On the other hand, the handling of fuzzy matrices covered determinants of fuzzy matrices, the eigenvalues of fuzzy matrices, and solving least-squares fuzzy linear equations. The approach demonstrated to be also applicable in a systematic way in handling new fuzzy probabilistic and statistical problems. This included extending the conventional probabilistic and statistical analysis for handling fuzzy random data. Application also covered the consolidity of fuzzy optimization problems. Various numerical examples solved have demonstrated that the new consolidity concept is highly effective in solving in a compact form the propagation of fuzziness in linear, nonlinear, multivariable and dynamic problems with different types of complexities. Finally, it is demonstrated that the implementation of the suggested fuzzy mathematics can be easily embedded within normal mathematics through building special fuzzy functions library inside the ***computational Matlab Toolbox*** or using other similar software languages.

1. Introduction

Consolidity [1–6] (which is a new introduced noun which means the ***act*** or ***quality*** of consolidation) is one of the inherent properties of the universe typically operating in ***fully fuzzy environment***[1]). It mainly measures the systems output reactions

[1] The term "***Fully Fuzzy Environment***" indicates an "***open Fully Fuzzy Environment***" defined as that all system parameters have fuzzy levels that can freely change all over the *positive* and *negative* values of the environment. A subclass of this environment is *bounded fuzzy environment* where all fuzzy levels can only change within restricted positive and negative ranges of the environment.

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versus combined input/system parameters' reactions when subjected to varying fuzzy environments or events (the ***system consolidity*** concept could be regarded as a general internal property of systems that can also be defined also far from fuzzy logic). System consolidity has been expressed by the consolidity index $F_{O/(I+S)}$ defined by the ratio $|F_O/F_{(I+S)}|$ where F_O denotes the system output factor (overall reaction) and $F_{(I+S)}$ designates the combined input and system parameters factor (overall action) [1–3]. The system is defined to be ***consolidated*** if $F_{O/(I+S)} < 1$ denoted by "***Class C***", ***neutrally consolidated*** if $F_{O/(I+S)} \approx 1$ designated by "***Class N***" and ***unconsolidated*** if $F_{O/(I+S)} > 1$ referred to as "***Class U***". A forth case is the mixed consolidated class denoted as "***Class M***" combining the feature of all the three classes.

Consolidity plays role in the parameters changes of all systems when subjected to events or varying environments ***"on and above"*** their normal operations or original set points[2] [5,6]. Its creation was based on magnificent physical laws that enables its consolidity regardless of all the ongoing changes and fuzzy occurrences that continuously take place. It is also an inherent property in most living species. For example for human beings, their built-in self healing mechanisms constitute one form of their surviving consolidity. Such consolidity, however, changes in their scale level from one human to another.

The applications of the ***Systems Consolidity Theory*** are versatile and open for both *natural* and *man-made* systems. Such new measure of the systems will open endless areas of research in many interdisciplinary and multidisciplinary subjects. Only very tiny parts of the applications are handled by the systems' consolidity theory till now, and the area is completely open for further development and work. Another important application is to incorporate the consolidity index within the change pathway change of real systems when subjected to varying environments, events or activities [5,6].

The success of the applications of systems' consolidity theory to some preliminary examples have led to the importance of further development of the new theory and conducting wide methodological experimentation to many other case studies in various disciplines. Examples of these successful applications are the HIV/AIDS Epidemic model; the infectious diseases spread model, the Prey–Predator Population model, the Arm Race model, the drug concentration model, etc. [1]. Nevertheless, any further extension or implementation is strongly tied to having a sound systematic know-how to be used as the tool to handle various classes of problems, including mathematical functions, matrices, probability and statistics analysis in fully fuzzy environment.

It is, therefore, the target of this paper to present in a systematic way the know-how for developing the system's consolidity supported with numerical examples and necessary explanations. Such implementation will shed light toward the introduction of a new generation of future superior systems that combine excellent functionalities and strong consolidity.

[2] ***Consolidity index*** is an important factor in scaling system parameter changes when subjected to events or varying environment. For instance, for all coming events at say event state μ which are "***on and above***" the system normal situation or stand will lead to consecutive changes of parameters. Such changes follow the general relationship at any event step μ as: Δ Parameter change (μ) = ***Function***[consolidity (μ), varying environment or event (μ)] [6]. Two important common cases in real life of such formulation are the ***linear***(or linearized) and the ***exponential*** relationship.

Similar approaches can also be devised for building *symbolic-based* libraries to cover wider classes of linear, nonlinear, multivariable and dynamic problems with different types of complexities. This is beside the *systematic* implementation of the above concept to many fuzzy applications such as functions of different dimensionalities and types, analytic geometry, vector analysis, formulas derivatives ad integrals, matrix operations, and functions of fuzzy matrices; thus, forming the basic core for the development of a *generalized fuzzy mathematics* necessary for performing various consolidity analyses, as elucidated in Fig. 1.

In Fig. 1, each branch topic will be handled separately, and the suggested fuzzy algebra for consolidity analysis will be applied to representative examples and case studies to form the corresponding element of the generalized fuzzy mathematics. The solution sought will be through neat mathematical closed form solutions far from the previously reported Taylor's series expansion for nonlinear functions representation.

2. Basics of the system consolidity

2.1. Basic definition of system consolidity

Systems can be classified according to consolidity into three categories as follows:[3] see Fig. 2, [1]:

(i) ***Consolidated systems*** or well connected, under hold, under grasp, well linked, robust or well joined systems.
(ii) ***Neutrally consolidated systems***.
(iii) ***Unconsolidated systems*** or weakly connected, separated, non-robust or isolated systems.

The analysis will be based on the Arithmetic fuzzy logic-based representation introduced in [5–10]. This Representation is based on expressing each parameter X by two components: X_o the deterministic equivalence, and X_f the fuzzy equivalence representing a small uncertainty or value tolerance in the parameter X. The term X_f is modeled by the formula: $X_f = f_r \ell_x X_o$ where f_r is the relative unit fuzziness (usually a certain small percentage; this means that the effective values of the fuzzy component are less than the main original deterministic problem), and ℓ_x is the corresponding fuzzy level. For the sake of simplicity f_r is omitted in the representation and the parameter X is expressed by the following pair $X = (X_o, \ell_x)$. The fuzzy operation based on the Arithmetic fuzzy logic-based representation technique is summarized in Table 1. It can be observed from Table 1 that the operations (+ and −), also (· and /) are similar in the Arithmetic fuzzy logic-based representation approach, which is not the case for the ***Conventional Fuzzy Theory*** [10].

It was shown that the suggested approach is identical to that of the Conventional Fuzzy Theory for addition and gives weighted average fuzziness results for the subtraction operations. Moreover, it yields similar results of multiplication and division operations after ignoring the second order relative variations terms. Proof of this analogy can be found in reference [10].

[3] The ***System Consolidity*** concept could be regarded as a general internal property of systems that can also be defined far from fuzzy logic such as by using rough sets.

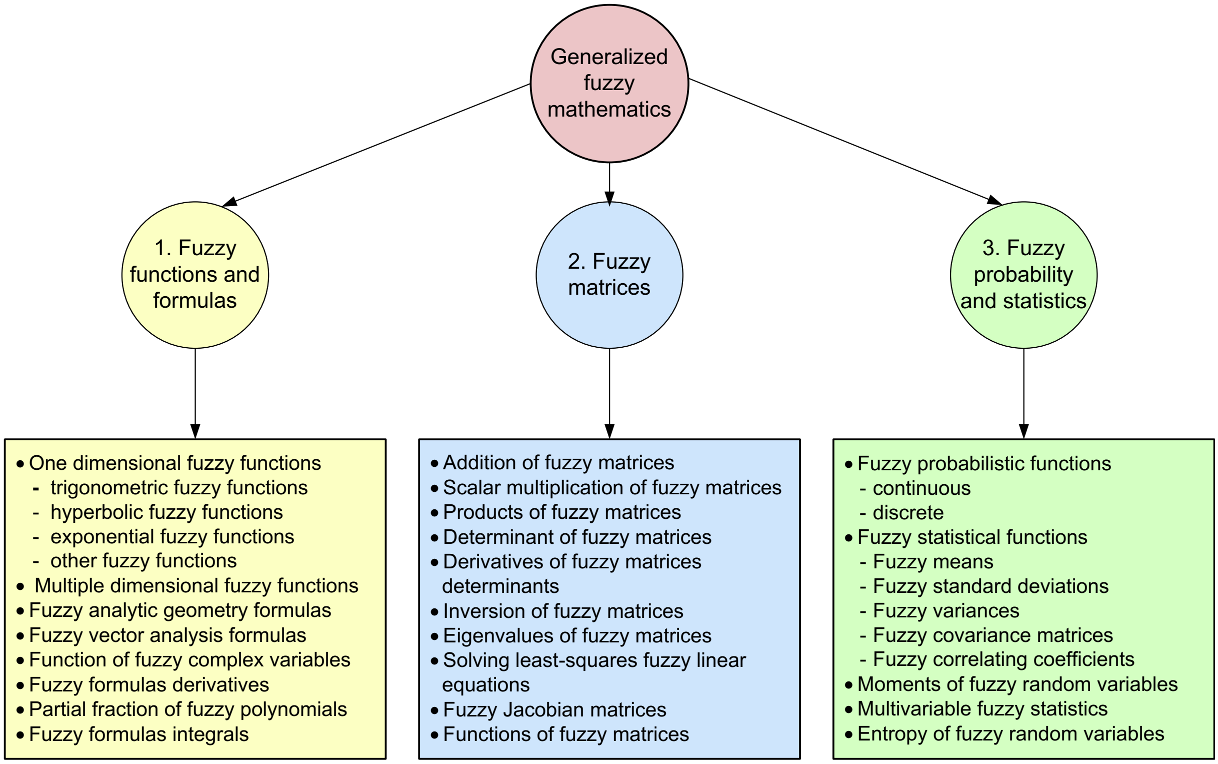

Figure 1 Development graph of the ***generalized fuzzy mathematics*** necessary for consolidity calculations and analysis.

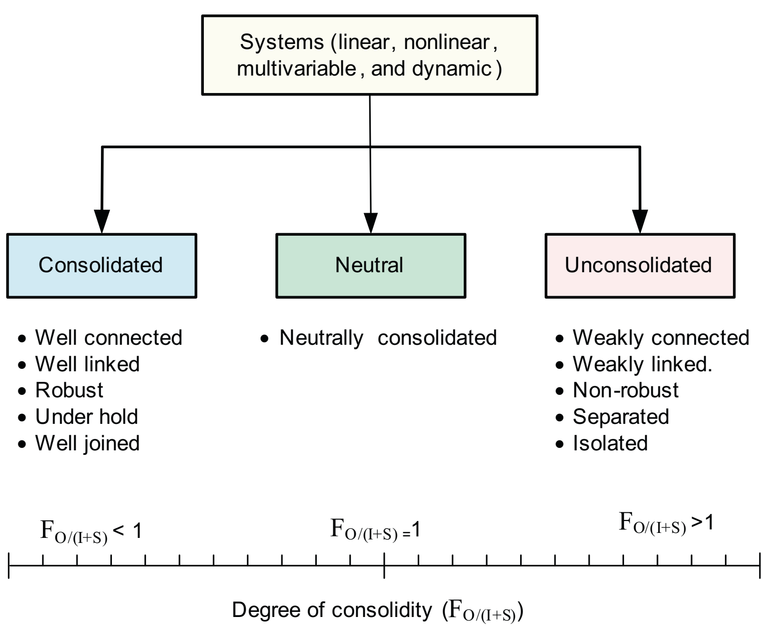

Figure 2 Basic definition of systems' consolidity.

A system operating at a certain stable original state in fully fuzzy environment is said to be ***consolidated*** if its overall output fuzziness is suppressed corresponding to their combined input and parameters fuzziness effect, and vice versa for ***unconsolidated*** systems. Neutrally consolidated systems correspond to marginal or balanced reaction of output fuzziness.

2.2. The system consolidity index

The ***system consolidity index*** is developed in this section. This index will measure the system overall output fuzziness behavior versus the combined input and system parameters variations. It describes the degree of how the systems react

Table 1 Summary of basic Arithmetic fuzzy logic-based representation algebra [7–9].

Name of operation	Symbolic representation of operation	Resulting values and fuzzy levels from operation
Addition	$X + Y$	$Z_o = X_o + Y_o$, and $\ell_z = \frac{\ell_x X_o + \ell_y Y_o}{X_o + Y_o}$
Subtraction	$X - Y$	$Z_o = X_o - Y_o$, and $\ell_z = \frac{\ell_x X_o - \ell_y Y_o}{X_o - Y_o}$
Multiplication	$X \cdot Y$	$X \cdot Y = (X_o Y_o,\ \ell_x + \ell_y)$
Division	X/Y	$X/Y = (X_o/Y_o,\ \ell_x - \ell_y)$

against input and system variation actions. Let us assume a general system operating in fully fuzzy environment, having the following elements:

Input parameters:

$$\underline{I} = (V_{I_i}, \ell_{I_i}) \tag{1}$$

such that $V_{I_i}, i = 1, 2, \ldots, m$ describe the deterministic value of input component I_i, and ℓ_{I_i} indicates its corresponding fuzzy level.

System parameters:

$$\underline{S} = (V_{S_j}, \ell_{S_j}) \tag{2}$$

such that $V_{S_j}, j = 1, 2, \ldots, n$ denote the deterministic value of system parameter S_j, and ℓ_{S_j} denotes its corresponding fuzzy level.

Output parameters:

$$\underline{O} = (V_{O_i}, \ell_{O_i}) \tag{3}$$

such that $V_{O_i}, i = 1, 2, \ldots, k$ designate the deterministic value of output component O_i, and ℓ_{O_i} designates its corresponding fuzzy level.

We will apply in this investigation, the weighted (or overall) fuzzy levels, first for the combined input and system parameters, and second for output parameters. As the relation between combined input and system with output is close to (or of the like type) of the multiplicative relations, the multiplication fuzziness property of Table 1 is applied for combining the fuzziness of input and system parameters.

For the combined input and system parameters, we have for the weighted fuzzy level to be denoted as the combined Input and System Fuzziness Factor F_{I+S}, given as:

$$F_{1+S} = \frac{\sum_{i=1}^{m} V_{I_i} \cdot \ell_{I_i}}{\sum_{i=1}^{m} V_{I_i}} + \frac{\sum_{j=1}^{n} V_{S_j} \cdot \ell_{S_j}}{\sum_{j=1}^{n} V_{S_j}} \tag{4}$$

Similarly, for the Output Fuzziness Factor F_O, we have

$$F_O = \frac{\sum_{i=1}^{k} V_{O_i} \cdot \ell_{O_i}}{\sum_{i=1}^{k} V_{O_i}} \tag{5}$$

Let the positive ratio $\lvert F_O/F_{I+S} \rvert$ defines the **system consolidity index**, to be denoted as $F_{O/(I+S)}$. Based on $F_{O/(I+S)}$ the system consolidity state can then be classified as [1–4]:

(i) ***Consolidated*** *if* $F_{O/(I+S)} < 1$, to be referred to as *"Class C"*.
(ii) ***Neutrally consolidated*** *if* $F_{O/(I+S)} \approx 1$, to be denoted by *"Class N"*.
(iii) ***Unconsolidated*** *if* $F_{O/(I+S)} > 1$, to be referred to as *"Class U"*.

For cases where the system consolidity indices lie at both consolidated and unconsolidated parts, the system consolidity will be designated as a mixed class or "*Class M*".

It must be pointed out that the same concept of consolidity index can be also applied in a *linguistic* rather than *numeric* type for descriptive systems that are not expressible in mathematical forms. This is an important aspect that can be considered in future research work.

For ***mixed consolidated/unconsolidated*** systems, we could face two special types of systems, namely:

(i) *Quasi-Consolidated Systems "Class* $\widetilde{C}$*":* These are mixed systems which are inclined more toward consolidation such as the center of gravity (Averaged value) has $F_{O/(I+S)}$ 1.
(ii) *Quasi-Unconsolidated Systems "Class* $\widetilde{U}$*":* These are mixed systems which are inclined more toward un-consolidation such as the center of gravity (Averaged value) has $F_{O/(I+S)} \gg 1$.

Table 2 Examples of derived consolidity indices for standard fuzzy mathematical functions at the original set-point x_0.

Ser.	Original function	Taylor's series expansion	Calculated compact form of fuzzy level	Consolidity index (symbolic form)
1	$y = \sin x$	$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$	$\ell_y = \ell_x \cdot x_0 \cdot \cos x_0/\sin x_0$	$\lvert x_0 \cos x_0/\sin x_0 \rvert$
2	$y = \cos x$	$y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$\ell_y = -\ell_x \cdot x_0 \cdot \sin x_0/\cos x_0$	$\lvert x_0 \sin x_0/\cos x_0 \rvert$
3	$y = \sinh x$	$y = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$	$\ell_y = \ell_x \cdot x_0 \cdot \cosh x_0/\sinh x_0$	$\lvert x_0 \cosh x_0/\sinh x_0 \rvert$
4	$y = \cosh x$	$y = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$	$\ell_y = \ell_x \cdot x_0 \cdot \sinh x_0/\cosh x_0$	$\lvert x_0 \sinh x_0/\cosh x_0 \rvert$
5	$y = \tanh^{-1} x$	$y = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$	$\ell_y = \ell_x \cdot x_0 \cdot (1 - x_0^2)^{-1}/\tanh^{-1} x_0$	$\lvert x_0 \cdot (1 - x_0^2)^{-1}/\tanh^{-1} x_0 \rvert$
6	$y = e^x$	$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot\cdot$	$\ell_y = \ell_x \cdot x_0$	$\lvert x_0 \rvert$
7	$y = e^x \sin x$	$y = x + x^2 + \frac{2x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} + \cdots$	$\ell_y = \ell_x \cdot x_0 \cdot (1 + \cos x_0/\sin x_0)$	$\lvert x_0\ (1 + \cos x_0/\sin x_0) \rvert$
8	$y = e^x \cos x$	$y = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$	$\ell_y = \ell_x \cdot x_0 \cdot (1 + \sin x_0/\cos\ x_0)$	$\lvert x_0 (1 + \sin x_0/\cos x_0) \rvert$
9	$y = e^{\tan x}$	$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3}{8}x^4 + \cdots$	$\ell_y = \ell_x \cdot x_0 \cdot (1 - \tan^2 x_0)$	$\lvert x_0 \cdot (1 - \tan^2 x_0) \rvert$
10	$y = \ln x$	$y = 2.\left(\frac{x-1}{x+1}\right) + \frac{2}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{2}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots$	$\ell_y = \ell_x/\ln x_0$	$\lvert 1/\ln x_0 \rvert$
11	$y = a^x = e^{x \ln a}$	$y = 1 + x \cdot \ln a + \frac{(x \cdot \ln a)^2}{2!} + \frac{(x \cdot \ln a)^3}{3!} + \cdot\cdot$	$\ell_y = \ell_x \cdot x_0 \cdot \ln a$	$\lvert x_0 \cdot \ln a \rvert$

We need therefore to develop soft computing-based algorithms for determining the external boundary of the consolidity signature, and its center of gravity. This step can be carried out using two different approaches. The first is to examine an exhaustive number of trials scanning a mesh of all possible input and system fuzziness, and then trace the external boundary or envelope of the results. A second approach is to build intelligent searching algorithm that attempts to allocate and follow up the external boundary of the consolidity signature (zone).

For some situations, it is possible for some functions to develop the compact mathematical form of the consolidity index $\lfloor F_O/F_{I+S} \rfloor$. Examples of these functions are shown in the next section.

3. System consolidity of fuzzy functions

3.1. One-dimensional fuzzy functions

In this section we will develop compact formulas for the system consolidity index of selected well known fuzzy functions such as the trigonometric, hyperbolic and exponential functions. The analysis starts by expressing each function by its equivalent of Taylor's series expansion.

In general, for the fuzzy series expressed as [11,12]:

$$f(x) = \sum_{i=0}^{\infty} a_i \, x^i \tag{6}$$

we have been using the Arithmetic fuzzy logic-based representation approach for the following corresponding fuzzy level:

$$\ell\{f(x)\} = \left\{ \sum_{i=0}^{\infty} a_i \cdot x_0^i \cdot i\ell_x \right\} \Big/ \sum_{i=0}^{\infty} a_i \cdot x_0^i \tag{7}$$

Applying formulas (6) and (7) to various selected function, it is easy to reach after some straightforward derivations the new compact form of their consolidity index as shown in Table 2.

Some numerical results for the consolidity of the functions provided in Table 2 is shown in Table 3 for some selected values of the fuzzy parameter x. The status of each consolidity index is shown also in the table.

Table 4 Consolidity results of selected three dimensional fuzzy functions.

Aspect	x	y	z	$u(x, y, z)$	$v(x, y, z)$	$w(x, y, z)$
Value	1	2	3	78	0.2673	1.0351
Fuzzy levels	5	3	4	2	−4	−1
	4	2	5	1	−4	−2
	2	3	4	3	−4	−1
	2	−3	−3	−4	3	1
	−4	−3	−6	6	5	2
	−6	5	4	9	−4	3
	4	−6	6	5	7	−3
	7	1	4	7	8	−1
	1	6	3	7	5	−1
Average calculated value of $F_{O/(I+S)}$				1.7073	1.6443	0.4724
Overall consolidity class				*M*	*M*	*M*

The results of the implementation of the consolidity theory to some selected standard functions indicated that their consolidity indices vary from consolidated to unconsolidated forms according to the various setting points selected for these functions.

It is remarked at this point that the derivation of the consolidity index of the standard functions in compact form represents a real impetus for pushing the new theory and will help in making its future implementation follows a neat and smooth path.

3.2. Multi-dimensional fuzzy functions

Let us consider the three dimensional functions of the three fuzzy variables x, y and z as:

$$u(x, y, x) = x \cdot y^2 \cdot z^3 - 5 \cdot x^2 \cdot y \cdot z \tag{8}$$

$$v(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} \tag{9}$$

and

$$w(x, y, z) = \sqrt{\frac{x + y + z - 1}{x^2 + y^2 + z^2 y - 1}} \tag{10}$$

Table 3 Consolidity index results with status for some selected fuzzy standard functions.

Function	Consolidity index of y for different values of x					
	0.1	0.3	0.5	0.7	0.9	1.1
$y = \sin x$	0.9967[a]	0.9698[a]	0.9152[a]	0.8311[a]	0.7142[a]	0.5599[a]
$y = \cos x$	0.0100[a]	0.0928[a]	0.2732[a]	0.5896[a]	1.1341 [b]	1.2817 [b]
$y = \sinh x$	1.0033 [b]	0.6492[a]	1.0820 [b]	1.1582 [b]	1.2565 [b]	1.3741 [b]
$y = \cosh x$	0.0100[a]	0.0874[a]	0.3022[a]	0.4231[a]	0.6447[a]	0.8805[a]
$y = e^x$	0.1000[a]	0.3000[a]	0.5000[a]	0.7000[a]	0.9000[a]	1.1000 [b]
$y = e^x \sin x$	1.0967 [b]	1.2698 [b]	1.4152 [b]	1.5311 [b]	1.6142 [b]	1.6599 [b]
$y = e^x \cos x$	0.1100[a]	0.3928[a]	0.7732[a]	1.2896 [b]	2.0341 [b]	3.2612 [b]
$y = e^{\tan x}$	0.0990[a]	0.2713[a]	0.3508[a]	0.2034[a]	0.5292[a]	3.1463 [b]
$y = \ln x$	0.4343[a]	0.8306[a]	1.4427 [b]	2.8037 [b]	9.4912 [b]	10.4921 [b]
$y = a^x$	0.0693[a]	0.2079[a]	0.3466[a]	0.4852[a]	0.6238[a]	0.7625[a]

[a] Means consolidated (*Class C*).
[b] Means unconsolidated (*Class U*).

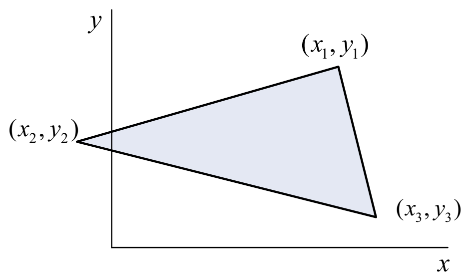

Figure 3 Sketch of triangle with three fuzzy vertices.

It must be observed that similarly the fuzzy level compact form of $\sqrt{f(x,y,z)}$ is approximately $\frac{1}{2}\cdot\ell\{f(x,y,z)\}$.

For some selected fuzzy scenarios of x, y and z in the positive scale, the corresponding fuzzy results of the above functions are summarized in Table 4. All over implementation procedure in this paper, the exact values of fuzzy levels are preserved all over the calculations and are rounded to integer values only at the final results. The results indicated that the consolidity of the three dimensional functions are of all of the mixed type, which is a combined consolidated and unconsolidated form.

3.3. Fuzzy analytic geometry

In this section, the fuzzy analytic geometry is handled by the suggested generalized fuzzy mathematics through solving a representable example [11].

Example: Area of triangle of fuzzy vertices

Consider the area of triangle with fuzzy matrices at the two dimensional vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3). The area can be expressed as, see Fig. 3:

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad (11)$$

$$A = \pm\frac{1}{2}(x_1y_2 + y_1x_3 + y_3x_2 - y_2x_3 - y_1x_2 - x_1y_3)$$

where the sign is chosen so that the area is nonnegative. The area is zero if the points all lie on one line.

Table 5 Consolidity results of the area of triangle with fuzzy vertices.

Aspect	x_1	y_1	x_2	y_2	x_3	y_3	A	$F_{O/(I+S)}$
Value	1	1	−2	−2	4	6	3	7.3333
Fuzzy levels	4	4	3	2	1	1	7	3.5556
	3	−3	2	−5	3	−2	3	1.5000
	2	4	2	4	5	3	−6	2.6667
	1	1	2	1	2	2	5	4.0000
	1	−3	2	4	3	1	−2	3.5556
	−1	−3	3	4	3	2	−1	1.2308
	5	4	2	3	2	1	2	7.3333
	3	2	5	4	7	3	9	2.1010
	4	1	3	6	7	2	−5	1.5802
Average calculated value of $F_{O/(I+S)}$								3.0581
Overall consolidity class								*U*

As a numerical example, we consider the various fuzzy scenarios of the vertices as shown in Table 5. The results reveal that the area of the fuzzy triangle is of the unconsolidated type. In general, there is no definition for the fuzziness of any complex variable as a whole, but the fuzzy concept can be implemented separately for both the ***real component*** and ***imaginary component***of the complex variable.

3.4. Fuzzy vector analysis

The fuzzy vector analysis using the generalized fuzzy mathematics will be demonstrated by solving several examples.

Example: Fuzzy volume of parallelepiped

The expression $A\cdot(B\times C)$ is in absolute value equal to the volume of a parallelepiped with sides A, B and C, as shown in Fig. 4.

If $A = A_1i + A_2j + A_3k, \quad B = B_1i + B_2j + B_3k,$ and $C = C_1i + C_2j + C_3k$ then

$$A\cdot(B\times C) = A\cdot\begin{vmatrix} i & j & k \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \quad (12)$$

$$\text{Volume} = A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)$$

$$V = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \quad (13)$$

As a numerical example we will consider the volume of parallelepiped with the sides shown in Table 6, and different corresponding fuzzy levels. The results using the generalized fuzzy mathematics are also shown in the same table. The results indicate that the consolidity of the volume of the fuzzy parallelepiped is of the quasi unconsolidated type. This is originally a mixed type, but the center of gravity of the consolidity zone is inclined more toward the unconsolidated side (Quasi-Unconsolidated type).

3.5. Functions of fuzzy complex variables

In this section, the suggested generalized fuzzy mathematics is extended to cover functions of fuzzy complex variables.

Example: Various fuzzy complex expressions

We will express each function of fuzzy variables x, y in the form $u(x, y) + i\,v(x, y)$, where u and v are real. The application of the system consolidity theory for such complex expressions will be seen to be also systematic, by considering the real part and the imaginary part separately during the calculation analysis.

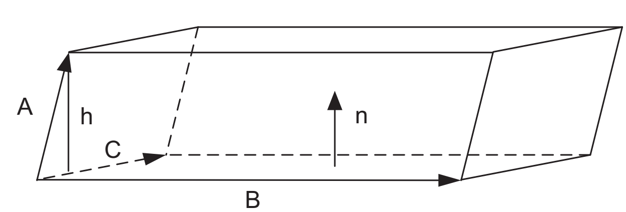

Figure 4 A sketch showing a general form of fuzzy parallelepiped.

Table 6 Consolidity results of the volume of fuzzy parallelepiped.

Aspect	A_1	A_2	A_3	B_1	B_2	B_3	C_1	C_2	C_3	V	$F_{O/(I+S)}$
Value	3	1	−1	20	1	2	1	5	7	30	
Fuzzy levels	7	1	5	4	3	1	4	2	1	−8	3.2804
	4	4	3	3	4	2	3	2	5	−6	1.7453
	2	3	2	1	3	1	1	2	1	−4	2.8000
	1	2	−3	2	1	3	2	−1	1	−2	1.9175
	−2	3	−5	1	3	−3	1	4	6	1	0.4156
	−5	−4	4	−5	3	3	−2	3	5	2	1.9833
	4	5	−3	2	2	5	−3	−3	3	−3	1.4539
	2	5	5	1	6	2	4	1	6	−2	0.5783
	3	1	2	6	6	4	1	1	2	−6	2.6444
Average value of $F_{O/(I+S)}$											1.8687
Overall consolidity class											$\widetilde{U}$

Table 7 Consolidity results of various functions of fuzzy complex variables.

Aspect	x	y	z^3		$1/(1-z)$		$\ln(z)$	
			u_1	v_1	u_2	v_2	u_3	v_3
Value	3	2	−9	46	−0.25	0.25	1.2825	0.5880
Fuzzy levels	3	2	1	8	7	5	2	−1
	1	2	11	4	1	2	1	1
	4	2	−4	11	10	6	3	−2
	−2	−1	2	−5	−5	−3	−1	1
	3	1	−7	8	8	5	2	−2
	4	3	4	11	9	6	3	−1
	1	1	3	3	2	2	1	0
	2	4	14	7	3	3	2	1
Average value of $F_{O/(I+S)}$			3.1293	3.2120	2.4116	1.6864	0.8384	0.5438
Overall consolidity class			$\tilde{U}$	U	M	M	C	M

Case 1:

$$z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3) = u_1 + iv_1 \tag{14}$$

where u_1 and v_1 are fuzzy variables.

Case 2:

$$\frac{1}{1-z} = \frac{1}{1-(x+iy)} = \frac{1-x+iy}{(1-x)^2+y^2} = \frac{(1-x)}{(1-x)^2+y^2} + i\frac{y}{(1-x)^2+y^2} = u_2 + iv_2 \tag{15}$$

Case 3:

$$\ln(z) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(y/x) = u_3 + iv_3 \tag{16}$$

The expressions for calculating the fuzzy levels in the corresponding compact form of u_3 and v_3 can be illustrated as $|y/x| \leqslant 1$:

$$\ell_{u_3} = \frac{\ell\{x^2+y^2\}}{\ln(x_0^2+y_0^2)} \tag{17}$$

and

$$\ell_{v_3} = \frac{(\ell_y - \ell_x)\cdot x_0y_0}{(x_0^2+y_0^2)\cdot\tan^{-1}(y_0/x_0)} \tag{18}$$

As a numerical example, we will consider various cases of fuzziness of x and y as shown in Table 7. The results using the generalized fuzzy mathematics are also shown in the same table. These results indicate that the real part and the imaginary part of the complex functions considered follow different types of consolidity.

3.6. Ordinary differentiation of fuzzy functions

In this section, the generalized fuzzy mathematics is implemented for handling problems of ordinary differentiation of fuzzy functions.

Example: Miscellaneous fuzzy differentiation

Case1:

For x is a fuzzy variable, let

$$y = \cosh(x^2 - 3x + 1) \tag{19}$$

then

Table 8 Steps followed in calculating y' and y''.

Step	Function	Corresponding fuzzy level
1	$u = x^2 - 3x + 1$	$\ell_u = (x^2\cdot 2\ell_x - 3x\cdot\ell_x)/u$
2	$v = \sinh(u)$	$\ell_v = \ell_u\cdot u\cdot\cosh u/\sinh u$
3	$w = \cosh(u)$	$\ell_w = \ell_u\cdot u\cdot\sinh u/\cosh u$
4	$z = (2x-3)^2$	$\ell_z = 2\cdot(2\cdot x\cdot\ell_x)/z$
5	$y'' = 2v + z\cdot w$	$\ell_{y'} = [2v\cdot\ell_v + z\cdot w(\ell_z+\ell_w)]/y''$

$$y' = \frac{\partial y}{\partial x} = (2x - 3)\sinh(x^2 - 3x + 1) \quad (20)$$

and

$$y'' = \frac{\partial^2 y}{\partial x^2} = 2\sinh(x^2 - 3x + 1) + (2x - 3)^2\cosh(x^2 - 3x + 1) \quad (21)$$

In calculating the fuzzy level of (21), the following steps can made as illustrated in Table 8.

Case 2:

Let us consider the formula

$$x^2y + y^3 = 2 \quad (22)$$

Differentiating (22) with respect to x, yields

$$y' = \frac{-2xy}{x^2 + 3xy^2} \quad (23)$$

Also we have

$$y'' = \frac{d(y')}{dx} = -\frac{(x^2 + 3y^2)\cdot(2xy' + 2y) - (2xy)\cdot(2x + 6yy')}{(x^2 + 3y^2)^2} \quad (24)$$

The corresponding fuzzy numerical results of the two equations using generalized fuzzy mathematics are shown in Table 9. The results of the example indicate variation of the consolidity index between the different classes.

3.7. Partial fraction of fuzzy polynomials

In this section, the suggested generalized fuzzy mathematics is applied for solving problems of partial fraction of fuzzy polynomials.

Consider the following fourth-order expression

$$f(x) = \frac{a_1x^3 + a_2x^2 + a_3x + a_4}{(x^2 + b_1)\cdot(x + b_2)^2} \quad (25)$$

such as a_1, a_2, a_3, a_4, b_1 and b_2 are fuzzy variables. The expression (25) can be represented in the following partial fraction form:

$$f(x) = \frac{Ax + B}{x^2 + b_1} + \frac{C}{x + b_2} + \frac{D}{(x + b_2)^2} \quad (26)$$

such as A, B, C and D are also fuzzy variables.

Equating coefficients of (25) and (26), we arrive at the following equations:

Table 9 Consolidity results of ordinary differentiation of fuzzy function.

Aspect	Case 1				Case 2			
	x	y	y'	y''	x	y	y'	y''
Value	0.6	1.0984	0.8178	2.6501	1	1	−0.5	−0.375
Fuzzy levels	5	2	9	−10	3	5	−3	−11
	4	2	7	−8	2	4	−3	−8
	3	−1	6	−6	1	2	−1	−4
	2	1	4	−4	−2	−1	1	7
	1	0	2	−2	−2	−6	4	8
	−6	−3	−11	12	4	1	−2	−14
	−7	−3	−13	−14	−3	7	−3	9
	8	4	15	−16	2	−7	3	6
	9	4	17	−19	3	6	−4	−11
Average of $F_{O/(I+S)}$		0.4460	1.8718	2.0610	Average of $F_{O/(I+S)}$		1.1375	4.5032
Consolidity class		C	U	U	Consolidity class		M	U

Table 10 Consolidity results of the partial fraction numerical example.

Parameter	a_1	a_2	a_3	a_4	b_1	b_2	A	B	C	D	$F_{O/(I+S)}$
Value	−3	9	−22	52	4	−2	1	2.5	−4	−2.5	
Fuzzy levels	5	3	4	6	4	3	−6	−7	2	−3	3.2053
	3	2	3	4	1	3	−4	−7	−1	−1	3.6980
	1	2	1	2	2	1	−6	−2	−1	1	1.1068
	1	−1	2	−1	−1	−1	5	7	2	−7	0.8636
	−1	2	−2	−3	−3	−1	−5	0	−2	7	4.0445
	−2	4	4	3	1	3	−12	−4	−1	−1	3.4636
	7	4	−2	1	2	5	7	4	7	7	3.9823
	2	1	−2	1	1	−4	13	7	5	9	5.6487
	4	5	5	6	5	3	−12	−4	0	2	2.0984
Average calculated value of $F_{O/(I+S)}$											3.1235
Overall consolidity class											$\tilde{U}$

$$
\begin{aligned}
x^3:&\quad a_1 = A + C\\
x^2:&\quad a_2 = B + 2b_2A + b_2C + D\\
x^1:&\quad a_3 = b_2^2A + 2b_2B + b_1C\\
x^0:&\quad a_4 = b_2^2B + b_1b_2C + b_1D
\end{aligned}
\tag{27}
$$

Eq. (27) can be solved numerically by the Gaussian–Jordan elimination method. The results of the selected example using generalized fuzzy mathematics for different input fuzzy scenarios are shown in Table 10. The results indicate that the partial fraction example is of the Quasi-Consolidated type. Similar treatment of the know-how for implementing the consolidity theory can be extended to wide classes of examples of algebra, geometry, trigonometry, topology, mechanics, etc.

4. System consolidity of fuzzy matrices

4.1. Determinant of fuzzy matrices

In this section, the system consolidity theory implementation know-how is shown for some selected fuzzy mathematical operations [13,14].

Consider the general form matrix $A \in R^{n\times n}$ expressed as:

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}
\tag{28}
$$

The determinant of A can be written as

$$
\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} m_{ij}
\tag{29}
$$

such as $(-1)^{i+j} m_{ij}$ is the ij the cofactor of a_{ij}. For the corresponding fuzzy value applying the suggested fuzzy algebra, we have

$$
\ell\{\det A\} = \sum_{j=1}^{n} (-1)^{i+j} [\ell(a_{ij}) + \ell(m_{ij})]
\tag{30}
$$

Numerical example: Determinant of Vandermonde fuzzy matrix.

We will consider the calculation of matrices with fuzzy element. As an example, let us introduce the Vandermonde fuzzy determinant defined as:

$$
\det V = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix} = \Pi_{1\leqslant i<j\leqslant n}(x_i - x_j)
\tag{31}
$$

such that $x_1, x_2, \ldots, x_n$ are fuzzy parameters. The problem is first solved using the Gaussian–Jordan Elimination technique using the generalized fuzzy mathematics applied to the original matrix. The results are then verified with the answer of the right hand side of (31).

Table 11 Consolidity results of Vandermonde determinant.

Aspect	x_1	x_2	x_3	x_4	det V	$F_{O/(I+S)}$
Value	1	2	3	4	12	1.2222
Fuzzy levels	5	6	3	1	−3	3.3611
	4	3	2	2	7	1.0614
	3	3	2	1	2	4.8333
	2	1	6	−1	−3	2.5972
	−4	−1	1	−2	−5	4.9500
	−3	−5	−3	−2	−6	1.2222
	−1	−3	−7	2	9	3.3611
	1	2	4	1	10	4.9762
	2	3	3	1	5	2.7065
Average value of $F_{O/(I+S)}$						3.2095
Overall consolidity class						U

The numerical example is selected for the case of $n = 4$. For this example, the results using the generalized fuzzy mathematics are given in Table 11. Alternatively, using the right hand side of (29), we have

$$
\det V = (x_{1_0} - x_{2_0})\cdot(x_{1_0} - x_{3_0})\cdot(x_{1_0} - x_{4_0})\cdot(x_{2_0} - x_{3_0})\cdot(x_{2_0} - x_{4_0})\cdot(x_{3_0} - x_{4_0})
\tag{32}
$$

and

$$
\ell\{\det V\} = \sum_{1\leqslant i<j\leqslant n} \frac{x_{i_0}\ell_{x_i} - x_{j_0}\ell_j}{x_{i_0} - x_{j_0}}
\tag{33}
$$

The results demonstrate that the determinant of the fuzzy Vandermonde matrix is of the unconsolidated type with considerable high average value of the consolidity index.

4.2. Eigenvalues of fuzzy matrices

Consider the linear system $\dot{x} = Ax$ with

$$
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^{2\times 2}
\tag{34}
$$

such as a, b, c and d are fuzzy parameters.

The characteristic polynomial of matrix A is given by:

$$
P_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)
\tag{35}
$$

The eigenvalues of A can be expressed as:

$$
\lambda_1 = \frac{a+d}{2} + \frac{\sqrt{(a+d)^2 - 4(ad-bc)}}{2}
\tag{36}
$$

and

$$
\lambda_2 = \frac{a+d}{2} - \frac{\sqrt{(a+d)^2 - 4(ad-bc)}}{2}
\tag{37}
$$

The parameters λ_1 and λ_2 are also fuzzy variables. This problem is demonstrated by a numerical example as shown in Table 12 for two cases of real and complex eigenvalues, solved using the generalized fuzzy mathematics. The results indicate that the consolidity of the eigenvalues of the example are of the mixed type combing both consolidated and unconsolidated values.

Table 12 Consolidity results selected eigenvalues of fuzzy matrices problem.

Aspect	a	b	c	d	λ_1	λ_2
Value	3	2	1	2	4	1
Case 1: Real fuzzy eigenvalues						
Fuzzy levels	4	2	4	3	6	−6
	3	2	3	3	5	−4
	2	1	2	1	3	−4
	5	−2	5	4	5	2
	1	1	−1	3	1	6
	1	6	−6	4	1	8
	−5	−3	3	1	−4	2
	3	2	5	4	6	−6
Average calculated value of $F_{O/(I+S)}$					1.4901	2.2556
Overall consolidity class					*M*	*M*

Aspect	a	b	c	d	$Re(\lambda_1)$	$Im(\lambda_1)$	$Re(\lambda_2)$	$Im(\lambda_2)$
Value	3	−2	1	2	2.5	1.3228	2.5	−1.3228
Case 2: Complex fuzzy eigenvalues								
Fuzzy levels	4	2	4	3	4	7	4	7
	3	2	3	3	3	6	3	6
	2	1	2	1	2	3	2	3
	5	−2	5	4	5	2	5	2
	1	1	−1	3	2	1	2	1
	1	6	−6	4	2	2	2	2
	−5	−3	3	1	−3	6	−3	6
	3	2	5	4	3	10	3	10
Average calculated value of $F_{O/(I+S)}$					1.1056	1.9531	1.1056	1.9531
Overall consolidity class					*M*	*M*	*M*	*M*

Table 13 Consolidity results of a selected least-square fuzzy linear equations problem.

Parameter	a_{11}	a_{12}	a_{21}	a_{22}	a_{31}	a_{32}	b_1	b_2	b_3	x_1	x_2
Value	1	2	2	−1	−1	2	1	2	3	0.6	0.8
Fuzzy levels	3	6	3	5	5	4	5	4	5	−5	5
	4	3	3	5	5	−3	3	4	4	7	1
	2	3	3	3	3	4	2	2	4	−6	3
	1	2	2	1	1	2	2	1	2	4	1
	1	2	−1	−2	−2	−1	−1	−3	−2	−8	−1
	−1	−2	−3	3	3	1	1	2	−2	−2	−6
	−3	−2	−1	1	1	−2	7	5	3	2	−1
	1	5	3	1	2	4	1	4	2	−3	−6
	3	3	4	4	5	5	3	4	5	−7	−4
Average calculated value of $F_{O/(I+S)}$										2.1943	1.1433
Overall consolidity class										*U*	*M*

4.3. Solving least-square fuzzy linear equations

Consider the inconsistent fuzzy linear system $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{38}$$

where all parameters are fuzzy variables.

The solution of (38) is normally expressed in the form of the least square equation:

$$x_0 = (A^T \cdot A)^{-1} A^T b = A^* b \tag{39}$$

where $(\cdot)^T$ indicates the transpose of $(\cdot)$ and A^* designates the pseudo or generalized inverse of the matrix A.

Eq. (39) is solved for a selected numerical example as shown in Table 13 with different input fuzzy levels scenarios. The results of the consolidity analysis indicate that the consolidity indices of both x_1 and x_2 are of the mixed type (combined consolidated and unconsolidated forms).

Such mathematical treatment can be also extended for the implementation of systems consolidity theory for inversions of fuzzy matrices, derivatives of matrices, fuzzy Jacobian matrices and functions of fuzzy matrices.

5. System consolidity of fuzzy optimization

The developed approach is now elaborated for the mathematical programming problems by simply using spreadsheet representation with ***Visual Basic Applications*** (VBA) programming. However, the approach is general and can be applied to other unlimited forms of representations and other known programming software such as Matlab.

In this section, the system consolidity theory is applied to a fuzzy nonlinear constrained optimization problem described as follows [15–17].

Solve the nonlinear optimization problem Minimize $G = ax^2 + by^2 + cz^2$ subject to $x \cdot y = d$ where a, b, c and d are fuzzy variables.

Define the Lagrangean function of the problem as:

$$L = ax^2 + by^2 + cz^2 + \lambda(xy - d) \tag{40}$$

and the Kuhn–Tucker conditions are

$$\text{(i)} \quad \frac{\partial L}{\partial x} = 2ax + \lambda y = 0 \tag{41}$$

$$\text{(ii)} \quad \frac{\partial L}{\partial y} = 2by + \lambda x = 0 \tag{42}$$

$$\text{(iii)} \quad \frac{\partial L}{\partial z} = 2cz = 0 \tag{43}$$

Multiplying (i) and (ii) by x and y respectively and using $xy = d$, gives

Table 14 Consolidity results of the nonlinear fuzzy optimization example.

Parameter	a	b	c	d	x	y	z	G	$F_{O/(I+S)}$
Value	1	4	16	1	$\sqrt{2}$	$1/\sqrt{2}$	0	4	
(i) First solution									
Fuzzy levels	3	4	1	3	1	2	0	7	3.7632
	−1	3	−3	5	3	3	0	6	2.6625
	−1	3	5	7	4	4	0	8	1.7959
	4	3	5	7	2	5	0	11	2.2427
	2	2	4	5	2	3	0	7	1.9494
	4	3	4	3	2	1	0	7	0.8049
	−3	2	−3	3	2	−2	0	−5	2.8171
	5	1	−5	6	−4	−3	0	−3	0.8571
Average calculated value of $F_{O/(I+S)}$									1.90123
Overall consolidity class									*M*
	1	4	16	1	$-\sqrt{2}$	$-1/\sqrt{2}$	0	4	
(ii) Second solution									
Fuzzy levels	3	4	1	3	1	2	0	7	3.7632
	−1	3	−3	5	3	3	0	6	2.6625
	−1	3	5	7	4	4	0	8	1.7959
	4	3	5	7	2	5	0	11	2.2427
	2	2	4	5	2	3	0	7	1.9494
	4	3	4	3	2	1	0	7	0.8049
	−3	2	−3	3	2	−2	0	−5	2.8171
	5	1	−5	6	−4	−3	0	−3	0.8571
Average calculated value of $F_{O/(I+S)}$									1.90123
Overall consolidity class									*M*

Table 15 Examples of standard fuzzy probability density functions analysis at their original set-points.

Ser.	Name	Fuzzy probability density function	Fuzzy mean (μ_0, ℓ_μ)	Fuzzy variance (V_0, ℓ_V)
1	Beta	$P_x = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} \cdot (1-x)^{\beta-1}$ $B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ $0 < x < 1$	$\mu_0 = \frac{\alpha_0}{\alpha_0+\beta_0}$ $\ell_\mu = \ell_\alpha - \ell_{\alpha+\beta}$	$V_0 = \frac{\alpha_0\beta_0}{(\alpha_0+\beta_0)^2\cdot(\alpha_0+\beta_0+1)}$ $\ell_v = \ell_\alpha + \ell_\beta - 2\ell_{\alpha+\beta} - \ell_{\alpha+\beta+1}$
2	Binomial (discrete type)	$f(x,n,p) = P(X=x)$ $= \binom{n}{x} \cdot p^x(1-p)^{n-x}$ $x = 0,1,2,\ldots n$	$\mu_0 = n \cdot p_0$ $\ell_\mu = \ell_p$	$V_0 = n \cdot p_0 \cdot (1-p_0)$ $\ell_V = \ell_p - \ell_p \cdot p_0/(1-p_0)$
3	Erlang	$P_x = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$ $0 < x, r = 1,2,\ldots$	$\mu_0 = r_0/\lambda_0$ $\ell_\mu = \ell_r - \ell_\lambda$	$V_0 = r_0/\lambda_0^2$ $\ell_v = \ell_r - 2\ell_\lambda$
4	Exponential	$P_x = \lambda e^{-\lambda x}$ $0 \leqslant x$ $0 < \lambda$	$\mu_0 = \frac{1}{\lambda_0}$ $\ell_\mu = -\ell_\lambda$	$V_0 = \frac{1}{\lambda_0^2}$ $\ell_v = -2\ell_\lambda$
5	Gamma	$P_x = \frac{\lambda x^{r-1} e^{-\lambda x}}{\Gamma(x)}$ $0 < x, \quad 0 < r, \quad 0 < \lambda$	$\mu_0 = r_0/\lambda_0$ $\ell_\mu = \ell_r - \ell_\lambda$	$V_0 = r_0/\lambda_0^2$ $\ell_v = \ell_r - 2\ell_\lambda$
6	Geometric	$(1-p)^{x-1}p,$ $x = 1,2,\ldots,0 \leqslant p < 1$	$\mu_0 = 1/p_0$ $\ell_\mu = -\ell_p$	$V_0 = (1-p_0)/p_0^2$ $\ell_v = -p_0 \cdot \ell_p/(1-p_0) - 2\cdot\ell_p$
7	Lognormal	$P_x = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(\frac{-[\ell_n(x)-\theta]^2}{2.\omega^2}\right)$	$\mu_0 = e^{\theta_0+\omega_0^2/2}$ $\ell_\mu = \ell_\theta \cdot \theta_0 + \ell_\omega \cdot \omega_0^2$	$V_0 = e^{2\theta_0+\omega_0^2}\left(e^{\omega_0^2}-1\right)$ $\ell_v = 2\ell_\theta \cdot \theta_0 + 2\ell_\omega \cdot \omega_0^2$
8	Negative binomial (discrete)	$P_x = \binom{x-1}{r-1}(1-p)^{x-r}p^r$ $x = r, r+1, r+2,\ldots,0 \leqslant p \leqslant 1$	$\mu_0 = r_0/p_0$ $\ell_\mu = \ell_r - \ell_p$	$V_0 = r_0(1-p_0)/p_0^2$ $\ell_v = \ell_r - p_0.\ell_p/(1-p_0) - 2.\ell_p$
9	Normal	$P_x = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-m}{\sigma}\right)^2}$	$\mu_0 = m_0$ $\ell_\mu = \ell_m$	$V_0 = \sigma_0^2$ $\ell_v = 2\ell_\sigma$
10	Poisson (discrete type)	$f(x;\lambda) = P(X=x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$ $x = 0,1,2,\ldots,0 < \lambda$	$\mu_0 = \lambda_0$ $\ell_\mu = \ell_\lambda$	$V_0 = \lambda_0$ $\ell_V = \ell_\lambda$
11	Uniform	$P_x = \frac{1}{b-a}$ $a \leqslant x \leqslant b$ $a < b$	$\mu_0 = \frac{(b_0+a_0)}{2}$ $\ell_\mu = \frac{\ell_b \cdot b_0 + \ell_a \cdot a_0}{(b_0+a_0)}$	$V_0 = \frac{(b_0-a_0)^2}{12}$ $\ell_v = 2(\ell_b \cdot b_0 - \ell_a \cdot a_0)/(b_0-a_0)$

$$2ax^2 = -\lambda d = 2by^2 = \frac{2bd^2}{x^2} \tag{44}$$

$$x^4 = \frac{bd^2}{a} \quad \text{thus} \quad x = \left(\frac{bd^2}{a}\right)^{1/4} \quad \text{and} \quad y = \frac{d}{x} \tag{45}$$

The results for the numeric example are shown in Table 14 for different scenarios of input fuzzy levels. From this table the results of the consolidity analysis of the nonlinear optimization problem reveal that the objective function G for both solutions is of the mixed type, containing both consolidated and unconsolidated index values.

6. System consolidity of fuzzy probability and statistics

6.1. Fuzzy probability functions

Most important applications in real life are dealing with fuzzy data. These data will lead to generating corresponding probability density functions with fuzzy coefficients. Some examples of these functions are given in Table 15 with corresponding fuzzy level of their means and variances [18–20]. The compact form of the derivation of the fuzzy means and variances will directly lead to having analogous compact form of their consolidity indices. Such compact form realization of the consolidity indices will represent another impetus for fostering the new theory in handling fuzzy probability and statistics problems.

Similar analysis can be generalized for multivariate probability density, distribution and conditional functions of various continuous or discrete types.

As a demonstration of an example of the fuzzy probability density, let us consider the normal distribution function with $\mu_0 = 1$ and $\sigma = 3$, and different scenarios of fuzzy levels. The fuzzy level of the normal probability density function can be expressed using the generalized fuzzy theory as

$$\ell\{p_x\} = \ell\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\cdot\left(\frac{x-\mu}{\sigma}\right)^2}\right) = \ell_\sigma \cdot \left[1 + 0.5\cdot\left(\frac{x-\mu_0}{\sigma_0}\right)^2 + 0.5\cdot\frac{\mu_0\cdot\ell_\mu}{\sigma_0}\cdot\left(\frac{x-\mu_0}{\sigma_0}\right)\right] \tag{46}$$

The distribution of the effect of input parameters fuzziness on the values of the probability density function is shown in Table 16. It could appear from the table, that the impact of input parameters fuzziness has different effect at different points of the probability density curve.

To get more insight to the fuzzy probability density curve, we will plot the results of Table 16 as shown in Fig. 5 using the visual fuzzy logic-based representation and the color codes given in Table 17 [21–24]. Such colors are based on the logical sequence of these colors in the Hue or wavelength circle describing the colors spectrum. The concept is an extension

Table 16 Scenarios of effect of input parameters fuzziness on the values of the normal probability density function ($\mu_0 = 1$ and $\sigma_0 = 3$).

x	$P_x = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$				
		Fuzzy levels of different scenarios			
	Value	($\ell_\mu = 2$, $\ell_\sigma = 1$)	($\ell_\mu = -2$, $\ell_\sigma = 1$)	($\ell_\mu = 3$, $\ell_\sigma = -1$)	($\ell_\mu = 1$, $\ell_\sigma = -1$)
−8	0.0015	3	4	−4	−4
−4	0.0332	2	3	−3	−3
−3	0.0547	1	2	−3	−2
−2	0.0807	1	2	−2	−1
−1	0.1065	1	1	−2	−1
0	0.1258	1	1	−1	−1
1	0.1329	1	1	−1	−1
2	0.1258	1	1	−1	−1
3	0.1065	1	1	−1	−1
4	0.0807	2	1	−1	−1
5	0.0547	2	1	−1	−1
6	0.0332	3	2	−1	−2
10	0.0015	4	3	−2	−3

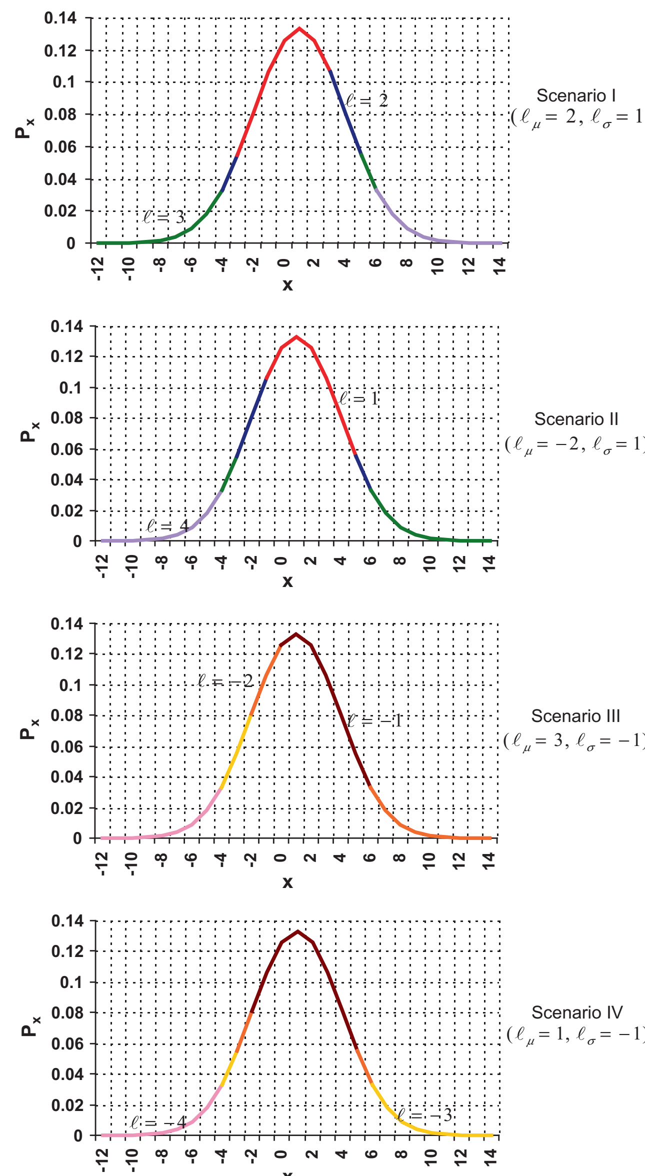

Figure 5 Sketches of effect of input parameters fuzziness on the values of the normal probability density function ($\mu_0 = 1$, $\sigma_0 = 3$).

Table 17 Definition of positive and negative colors sample scale for visual representations [22–24].

Ser.	Color	Color Code	RBG Color Index	Excel Color Index	Corresponding Fuzzy Level	Type
1		Green Light	(204,255,204)	35	+6	Positive colors
2		Blue Light	(153,204,255)	37	+5	
3		Violent (Lavender Light)	(204,153,255)	39	+4	
4		Green	(51,153,102)	50	+3	
5		Blue	(0,102,204)	5	+2	
6		Violent (Lavender)	(255,0,255)	7	+1	
7		Black	(0,0,0)	2	0	0
8		Red	(255,0,0)	3	-1	Negative colors
9		Orange	(255,102,0)	46	-2	
10		Yellow	(255,255,0)	27	-3	
11		Red Light	(255,153,204)	38	-4	
12		Orange Light	(255,153,0)	45	-5	
13		Yellow Light	(255,255,204)	19	-6	

of the well-known fuzziness similarity with ***gray scale*** to more ***general color scales.***

The plots show that the normal probability density function is divided into several zones of different fuzzy levels that are symmetrical around the mean.

For the sake of consolidity analysis of the normal probability density functions of fuzzy parameters m and σ of Table 15, we can see that for the mean we have $\ell_\mu/\ell_m = 1$ (Neutrally consolidated Index), and for the variance $V = \sigma_0^2$ we have $\ell_v/\ell_\sigma = 2$, which represents an unconsolidated variable.

6.2. Fuzzy statistical functions

In this section, the generalized fuzzy mathematics is applied for solving fuzzy statistical functions. Let x_i, $i = 1, 2, \ldots, n$, be a fuzzy sequence, then for the fuzzy mean of the sequence, we have

$$\mu_0 = \frac{1}{n}\sum_{i=1}^{n} x_{i_0} \tag{47}$$

and

$$\ell_\mu = \frac{\sum_{i=1}^{n} \ell_{x_{i_0}} \cdot x_{i_0}}{\sum_{i=1}^{n} x_{i_0}} \tag{48}$$

The corresponding variance is given by

$$V_0 = \frac{1}{n}\sum_{i=1}^{n} x_{i_0}^2 \tag{49}$$

and

$$\ell_v = \frac{\sum_{i=1}^{n} 2 \cdot \ell_{x_{i_0}} \cdot x_{i_0}^2}{\sum_{i=1}^{n} x_{i_0}} \tag{50}$$

Consolidity tests were carried out for the mean and variance of the four selected probability density functions of the Uniform, Lognormal, Gamma and Beta and the results are shown in Table 18. Once more in the implementation procedure, the exact values of fuzzy levels are preserved all over the calculations and are rounded to integer values only at the final results.

The results indicate that the consolidity indices of the mean μ and the variance V of the selected probability density functions are of different types belonging to different consolidity zones.

Such mathematical treatment can also be extended for the implementation of system consolidity theory to other fuzzy statistical functions such as correlation and covariance matrices, moments of fuzzy random variables, multivariable fuzzy statistics, and entropy of fuzzy random variables.

7. Building Matlab Toolbox library for system consolidity calculations

It follows from the above analysis that the presented fuzzy know-how could be an excellent tool to determine the propagation of fuzziness for various functions and matrices in fully fuzzy environment. It also could appear from the study how the corresponding fuzzy levels of various standard functions can be calculated systematically in a compact form. This will give the real need for building a fuzzy functions library to calculate within the course of calculations corresponding fuzzy levels.

Thus the research work could be directed toward transferring all basic fuzzy operations, fuzzy functions and matrices operations, as well as fuzzy statistical operations as built-in function in special computational Toolbox in Matlab or to be created as special functions inside other software languages. The building of such library will strengthen the capability of the generalized fuzzy mathematics approach to effectively handle fuzzy problems expressed and manipulated in matrix form regardless of their dimensionalities and types of operation; also for handling fuzzy data expressed by fuzzy probabilistic or statistical functions. Example of tentative work toward building such Matlab Library for helping in performing system consolidity analysis is shown in Table 19.

Similar approaches can also be devised for building libraries to cover other problems such as fuzzy multivariate

Table 18 Consolidity results of the fuzzy mean and variance of some selected probability density functions.

Aspect	a	b	μ	V	$F_{O/(I+S)}$	
Value	1	7	4	3	μ	V
(i) Fuzzy uniform probability density function						
Fuzzy levels	1	2	4	3	2.0000	2.3111
	2	1	2	4	2.0000	1.4815
	3	2	4	2	2.0000	1.7255
	−3	2	3	4	2.0000	4.1212
	2	−2	−3	−5	2.0000	3.5555
	−4	−3	−6	−6	2.0000	1.8133
	5	2	5	3	2.0000	1.2632
	4	8	15	17	2.0000	2.3111
	7	3	7	5	2.0000	1.3333
	1	6	11	14	2.0000	2.5426
Average value of $F_{O/(I+S)}$					2.0000	2.2458
Overall consolidity class					*U*	*U*
	θ	w	μ	V	$F_{O/(I+S)}$	
	1	.5	3.0802	7.9328	μ	V
(ii) Fuzzy lognormal probability density function						
Fuzzy levels	1	2	1	3	0.4870	2.2501
	2	1	1	5	0.4870	2.6999
	3	2	1	7	0.4870	2.6250
	−3	2	−1	−5	0.4870	3.7501
	2	−2	0	3	0.4870	4.4998
	−4	−3	−2	−10	0.4870	2.5909
	5	2	2	11	0.4870	2.7500
	4	8	3	12	0.4870	2.2500
	7	3	3	16	0.4870	2.7353
	1	6	1	5	0.4870	1.8750
Average value of $F_{O/(I+S)}$					0.4870	2.8026
Overall consolidity class					*C*	*U*
	r	λ	μ	V	$F_{O/(I+S)}$	
	1	0.5	2	4	μ	V
(iii) Fuzzy gamma probability density function						
Fuzzy levels	1	2	2	−3	1.5000	2.2501
	2	1	1	0	0.6000	0.0000
	3	2	1	−1	0.3750	0.3750
	−3	2	−5	−7	3.7501	5.2501
	2	−2	4	6	5.9997	8.9996
	−4	−3	−1	2	0.2727	0.5454
	5	2	2	3	0.7500	0.7500
	4	8	3	1	0.7500	0.2500
	7	3	−4	−12	0.7059	2.2500
	1	6	−5	−11	1.8750	0.1765
Average value of $F_{O/(I+S)}$					1.6578	2.4222
Overall consolidity class					*M*	*M*
	α	β	μ	V	$F_{O/(I+S)}$	
	1	0.5	0.6667	0.0889	μ	V
(iv) Fuzzy beta density function						
Fuzzy levels	1	2	1	−4	0.7500	2.7501
	2	1	1	−5	0.6000	3.1999
	3	2	2	−8	0.6250	3.1251
	−3	2	0	−6	0.2500	4.2501
	2	−2	0	3	0.0000	4.9997
	−4	−3	−2	11	0.6363	3.0909
	5	2	2	13	0.5833	3.2500
	4	8	4	−15	0.7500	2.7502
	7	3	3	−18	0.5882	3.2353
Average value of $F_{O/(I+S)}$					0.4783	3.0651
Overall consolidity class					*C*	*U*

Table 19 Some selected symbolic-based toolbox examples of the fuzzy functions calculations library at the original set-point x_0.

Ser.	Original function	Calculated symbolic-based fuzzy level	Proposed toolbox function of fuzzy level
1	$y = \sin x$	$\ell_y = \ell_x \cdot x_0 \cdot \cos x_0/\sin x_0$	$\ell_y = F\sin(x_0, \ell_x)$
2	$y = \cos x$	$\ell_y = -\ell_x \cdot x_0 \cdot \sin x_0/\cos x_0$	$\ell_y = F\cos(x_0, \ell_x)$
3	$y = \tan x$	$\ell_y = \ell_x \cdot x_0 \cdot \cos x_0/\sin x_0 - \ell_x \cdot x_0 \cdot \sin x_0/\cos x_0$	$\ell_y = F\tan(x_0, \ell_x)$
4	$y = \sinh x$	$\ell_y = \ell_x \cdot x_0 \cdot \cosh x_0/\sinh x_0$	$\ell_y = F\sinh(x_0, \ell_x)$
5	$y = \cosh x$	$\ell_y = \ell_x \cdot x_0 \cdot \sinh x_0/\cosh x_0$	$\ell_y = F\cosh(x_0, \ell_x)$
6	$y = \tanh x$	$\ell_y = \ell_x \cdot x_0 \cdot \cos x_0/\sin x_0 + \ell_x \cdot x_0 \cdot \sinh x_0/\cosh x_0$	$\ell_y = F\tanh(x_0, \ell_x)$
7	$y = \tanh^{-1} x$	$\ell_y = \ell_x \cdot x_0 \cdot \left(1 - x_0^2\right)^{-1}/\tanh^{-1} x_0$	$\ell_y = Fa\tanh(x_0, \ell_x)$
8	$y = e^x$	$\ell_y = \ell_x \cdot x_0$	$\ell_y = F\exp(x_0, \ell_x)$
9	$y = \ln x$	$\ell_y = \ell_x/\ln x_0$	$\ell_y = F\ln(x_0, \ell_x)$
10	$y = e^{\tan x}$	$\ell_y = \ell_x \cdot x_0 \cdot (1 + \tan^2 x_0)$	$\ell_y = Fe\ \tan(x_0, \ell_x)$
11	$y = e^x \sin x$	$\ell_y = \ell_x \cdot x_0 \cdot (1 + \cos x_0/\sin x_0)$	$\ell_y = F\ \exp(x_0, \ell_x) \cdot F\sin\ (x_0, \ell_x)$

Table 20 Some suggested areas of applications of the consolidity theory [1–3,30].

Basic sciences	Evolutionary systems	Engineering	Biology and medicine	Economics and finance	Political and management sciences	Social sciences and humanities[a]
Mathematics	Evolution theory	Control and robotics	Genetics	Financial systems	Political theory	Social science
Physics	Evolutionary models	Industrial systems	Bio-statistics	Econometrics	Behavior science	Literature
Chemistry	Global modeling	Aeronautics and space	Bioinformatics	Business	Management models	Communication studies
Biology	Global optimization	Chemical processes	Medicine	Commerce	Operations research	Psychology
Astronomy		Nuclear engineering	Biomedical engineering	Accounting	Organizations	Philosophy
Geology		Aerospace engineering	Pharmacology	Marketing	Development studies	Education
		Material engineering	Ecology	Operation management		Law

[a] The treatment in each discipline could be carried out either in *numeric* or *linguistic* type based on the considered nature of system's representation.

regression analysis, fuzzy Fourier Transform, and spectra analysis of fuzzy random variables of different dimensionalities. This will open the door in the future toward the solving of many previously forbidden classes of real life problems in fully fuzzy environments.

As an example of using the above special fuzzy function, let us introduce y such as

$$y = e^{-ax} \cdot \sin bx \cdot \ln cx/\sec dx \tag{51}$$

Then, the fuzzy level of y using the above fuzzy functions library can be expressed as:

$$\ell_y = F\exp(-a \cdot x_0, \ell_x) + F\sin(b \cdot x_0, \ell_x) + F\ln(c \cdot x_0, \ell_x) - F\cos(d \cdot x_0, \ell_x) \tag{52}$$

For the function of (52), the consolidity index can easily be expressed as $\lfloor \ell_y/\ell_x \rfloor$ which will follow directly from substitution.

All basic fuzzy operations, fuzzy functions and matrices operations, as well as fuzzy statistical operations can be built as a special library of the ***computational Matlab Toolbox***or can be created as special functions in other software languages [25].

8. Discussions of the applications of system consolidity

With the presentation in a systematic way of the fuzzy know-how in this paper, the road is now paved to start examining the system consolidity of existing natural and man-made systems. Moreover, such fuzzy know-how will be indispensable for checking (and double checking) open spectrum of future applications during their modeling, analysis and design stages [26–29].

The applications of the consolidity theory cover almost facets of existing sciences. A brief account of these applications is provided in Table 20 [1–3,30]. In general, consolidity is an internal property of systems that enables giving an in-depth look inside such systems, regardless of their field of applications. Such property will lead to giving a new forum for better understanding of various sciences. With the developed know-how for consolidity calculations, new classes of advanced systems with strong consolidity will be born and will be taken for granted as the future standard of systems in various disciplines.

Using the presented know-how for the calculations of system consolidity, researchers, designers and developers are now in excellent position to start building new generation of systems with strong consolidity standards. In the same time, they should start searching within the existing *natural* and *man-made* systems for ways to keep them always in good consolidated states. Though these missions look very challenging, yet the consolidity theory know-how presented in this paper could help in crossing *quickly* a substantial span of such challenge.

For the implementation of ***consolidity theory*** to existing man-made systems, the situation could be possible by altering parameters of the system within the utmost extend permitted for changes. As for ***natural systems***, the system consolidity improvement matter could also be possible by interfering within the system parameters together with environment and trying to direct the physical process toward better targeted consolidity. Moreover, for the implementation of the consolidity approach for ***man-made systems***, it is possible that various prototypes can be designed fulfilling almost the same degree of functionality. These systems can be ranked starting from the best consolidated one with the lowest consolidity index score (the superior consolidated prototype).

9. Conclusions

The paper presented ***a comprehensive analysis*** of the necessary know-how for developing the system consolidity theory for various basic fuzzy mathematical problems. The problem of formulating system consolidity theory was extended in this paper to cover general classes of fuzzy mathematical functions, matrices and statistics. It is shown that the system consolidity index can be expressed in compact forms for most standard functions such as trigonometric, hyperbolic and exponential functions. Moreover, the consolidity approach is highly applicable to fuzzy problems expressed and manipulated in matrix form regardless of their dimensionalities and types of operation or to fuzzy data expressed by fuzzy probabilistic or statistical functions. The technique can also be applied smoothly to cover most of the basic fuzzy probabilistic and statistics functions and operations. Extension of the consolidity theory know-how was also elucidated for handling fuzzy optimization problems of the linear and nonlinear types.

The results of the system consolidity theory give rise to building a comprehensive library for calculating the corresponding propagated fuzziness for these functions. Therefore, the generalized fuzzy mathematics can be easily embedded with the conventional mathematics through incorporating such fuzzy functions library as a special ***computational Matlab Toolbox***, or through creating special fuzzy-based functions in other software packages. These cover building a comprehensive fuzzy-based library to accommodate all fuzzy functions, matrices and statistical operations. In all cases, the proposed fuzzy know-how follows the conventional mathematics and statistics and can be extended easels to many other branches. Extension is possible to other mathematical categories of algebra, geometry and topology, calculus, dynamics, mechanics, etc. System consolidity index can be also implemented in a *linguistic* rather than *numeric* type for descriptive systems that are not expressible in mathematical forms.

The presentation of the know-how in this paper will open the door toward future solving many previously forbidden classes of real life system consolidity problems in fully fuzzy environments. Examples of some disciplines that could benefit from the presented know-how are the fields of basic sciences, evolutionary systems, engineering, astronomy, life sciences, environmental sciences, ecology, biology, medicine, economics, finance, political and management sciences, social science, communication studies, humanities, and education. Methodological development and field experimentation of this new system consolidity theory are thus recommended for solving many applications in these disciplines including all the cycles of modeling, analysis, optimization and design in fully fuzzy environment.

In conclusion, it is speculated that with the wide use of the proposed know-how, the implementation of the system consolidity theory will uncover many unanswered postponed intriguing questions about the malfunction and collapse of some of our existing systems due to their inferior consolidity. On the other hand, researchers and developers with the right know-how tools in their hands should seek building a new generation of superior systems to be designed on the basis of *excellent* functionality and *strong* consolidity.

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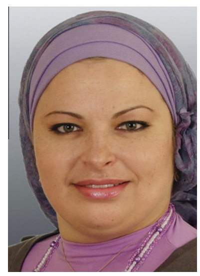

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